Classifying spaces for families for systolic and small cancellation groups

Tomasz Prytuła
University of Copenhagen

UNIVERSITY OF COPENHAGEN

Introduction

We construct a finite-dimensional model for $\underline{\underline{E}}G$ where G is a systolic group. Our approach parallels the one used for CAT(0) groups, yet some exotic properties of systolic complexes are exploited in order to give better dimension bounds.
We also construct low-dimensional models for $\underline{E}G$ and $\underline{\underline{E}}G$ for graphical small cancellation groups.
This is joint work with Damian Osajda.

Classifying spaces for families

Let G be a group and $\mathcal{F}$ a family of its subgroups. A *model for classifying space of G for the family $\mathcal{F}$* is a G–CW-complex $E_{\mathcal{F}}G$ such that for every subgroup H of G the fixed point set

$$(E_{\mathcal{F}}G)^H = \begin{cases} \text{contractible} & \text{if } H \in \mathcal{F}, \\ \text{empty} & \text{otherwise.} \end{cases}$$

Families we consider:

1. $\mathcal{FIN}$: the family of all finite subgroups of G.
2. $\mathcal{VCY}$: the family of all virtually cyclic subgroups of G.

Let $\underline{E}G := E_{\mathcal{FIN}}G$ and $\underline{\underline{E}}G := E_{\mathcal{VCY}}G$.

An importance of $\underline{\underline{E}}G$ comes from the *Farrell-Jones conjecture*: the 'assembly map'

$$\mathcal{H}^G_*(\underline{\underline{E}}G, K_R) \to K_*(R[G])$$

is an isomorphism.
To compute the left-hand side we need simple models for $\underline{\underline{E}}G$.

Systolic complexes

A simply connected simplicial complex X is *systolic* if it is flag and if every cycle in X of length less than 6 has a diagonal ([Che00], [JŚ06]).

Features of systolic complexes:

1. defined by an easily checkable, combinatorial condition,
2. simplicial analogues of non-positively curved metric spaces,
3. lead to a rich class of groups having exotic properties.

Theorem (Osajda-P., 2015)

Let a group G act properly on an n–dimensional systolic complex X.
Then there exists a G–CW-model for $\underline{\underline{E}}G$ of dimension

$$\dim \underline{\underline{E}}G = \begin{cases} n+1 & \text{if } n \leqslant 3, \\ n & \text{if } n \geqslant 4. \end{cases}$$

Method

We use a push-out construction of Lück and Weiermann [LW12]. The idea is to glue cells equivariantly to the model for $\underline{E}G$, in order to create contractible fixed point sets for infinite virtually cyclic subgroups.

- The systolic complex X is a model for $\underline{E}G$ ([CO15]).
- The cells we glue to X are related to hyperbolic isometries of X.

Tools

1. hereditary asphericity properties of systolic complexes
2. hyperbolic isometries and their minimal displacement sets
3. quasi-isometric rigidity results

Hyperbolic isometries

An isometry h of a systolic complex X having no fixed points is called *hyperbolic*. Let

$$\mathrm{Min}(h) = \mathrm{span}\{x \in X^{(0)} \mid d(x, h(x)) \text{ is minimal}\}.$$

The following theorem is the crucial step in our construction.

Theorem (Osajda-P., 2015)

Let h be a hyperbolic isometry of a systolic complex X. Then there is a quasi-isometry

$$c\colon T \times \mathbb{R} \to \mathrm{Min}(h)$$

where T is a simplicial tree.

Graphical small cancellation

Let $X^{(1)}$ be a graph and consider a family of graphs $\{\Gamma_i\}_{i \in I}$ together with combinatorial maps $\phi_i \colon \Gamma_i \to X^{(1)}$. A *graphical 2–complex* is a "coned-off" space

$$X = X^{(1)} \cup_{(\phi_i)} \bigcup_i \operatorname{cone} \Gamma_i$$

formed by gluing to $X^{(1)}$ a topological cone along each graph Γ_i.

- A *piece* in X is a path $P \to X^{(1)}$ that is contained in the images of 2 distinct graphs Γ_i and Γ_j.
- A graphical 2–complex X satisfies $C(6)$ *small cancellation condition* if no cycle $C \to X^{(1)}$ is a concatenation of less than 6 pieces.

Systolic duals of $C(6)$ complexes

Let X be a simply connected graphical 2–complex. Consider its covering by the cones $\{\operatorname{cone} \Gamma_i\}_{i \in I}$. Define the simplicial complex

$$\hat{X} = \mathcal{N}(\{\operatorname{cone} \Gamma_i\}_{i \in I})$$

as the *nerve* of this covering. We call $\hat{X}$ the *dual* of X.

We show that if X satisfies $C(6)$ condition, then $\hat{X}$ is systolic. This is a generalization of a theorem of D. Wise [Wis] for classical small cancellation complexes. As a corollary we obtain the following.

Theorem (Osajda-P., 2015)

Let a group G act properly on a $C(6)$ graphical small cancellation complex X. Then:

- The complex X is a 2–dimensional model for $\underline{E}G$.
- There exists a G–CW-model for $\underline{\underline{E}}G$ of dimension at most 3.

References & Contact Information

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t.prytula@math.ku.dk, tomaszprytula.com